



## Lecture 6: Seifert-van Kampen Theorem



Seifert-van Kampen Theorem gives a very effective method to compute fundamental groups of a space from its building blocks.



## Theorem (Seifert-van Kampen Theorem, Groupoid version)

Let  $X = U \cup V$  where  $U, V \subset X$  are open. Then the diagram

$$\begin{array}{ccc} \Pi(U \cap V) & \longrightarrow & \Pi(U) \\ \downarrow & & \downarrow \\ \Pi(V) & \longrightarrow & \Pi(X) \end{array}$$

is a pushout in the category Groupoid.







**Uniqueness:** Let  $\gamma: I \rightarrow X$  be a path in  $X$  with  $x_t = \gamma(t)$ . We subdivide  $I$  (by its compactness) into

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that  $\gamma_i := \gamma(t_{i-1}, t_i)$  lies entirely in  $U$  or  $V$ . Then

$$F([\gamma]) = F([\gamma_m]) \cdots F([\gamma_1])$$

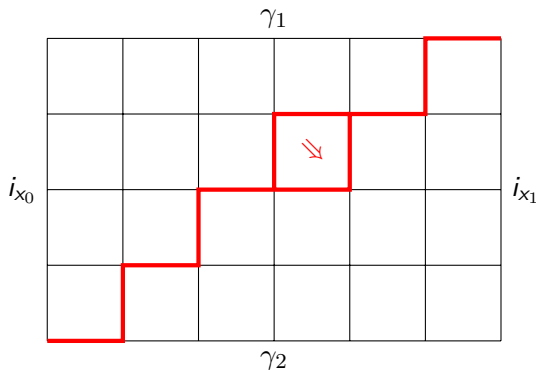
is determined uniquely in  $\mathcal{C}$  as each term is.

**Existence:** Given a path  $\gamma$ , we can define  $F([\gamma])$  using a subdivision of  $\gamma$ , where the result does not depend on the choice of the subdivision (by the definition of pushout). We need to show that this is well-defined on homotopy class.



This follows from a refined double subdivision of  $I \times I$ , as shown in the picture below. Each square represents a homotopy lying entirely in either  $U$  or  $V$  and combining them together gives the required homotopy.

$$F(\gamma_1) \simeq F(\gamma_1 \star i_{x_0}) \simeq F(i_{x_1} \star \gamma_2) \simeq F(\gamma_2)$$





## Theorem (Seifert-van Kampen Theorem, Group version)

Let  $X = U \cup V$  where  $U, V \subset X$  are open and  $U, V, U \cap V$  are path connected. Let  $x_0 \in U \cap V$ . Then the following diagram

$$\begin{array}{ccc}
 \pi_1(U \cap V, x_0) & \longrightarrow & \pi_1(U, x_0) \\
 \downarrow & & \downarrow \\
 \pi_1(V, x_0) & \longrightarrow & \pi_1(X, x_0)
 \end{array}$$

is a pushout in the category Group.



# Proof

Denote by  $\underline{G}$  the groupoid with one object for a group  $G$ .

For each  $x \in X$ , we fix a choice of

$$[\gamma_x] \in \text{Hom}(x_0, x)$$

such that  $\gamma_x$  lies entirely in  $U$  when  $x \in U$  and  $\gamma_x$  lies entirely in  $V$  when  $x \in V$ . Note that  $\gamma_x$  lies entirely in  $U \cap V$  when  $x \in U \cap V$ . This can be achieved since  $U, V, U \cap V$  are all path connected.



Consider the following functors

$$\Pi_1(U) \rightarrow \underline{\pi_1(U, x_0)}$$

$$\Pi_1(V) \rightarrow \underline{\pi_1(V, x_0)}$$

$$\Pi_1(U \cap V) \rightarrow \underline{\pi_1(U \cap V, x_0)}$$

$$\gamma \mapsto \gamma_{x_2}^{-1} \star \gamma \star \gamma_{x_1}, \quad \gamma \in \text{Hom}(x_1, x_2).$$

These functors are all retracts in Groupoid, in other words,

$$\underline{\pi_1(U, x_0)} \rightarrow \Pi_1(U) \rightarrow \underline{\pi_1(U, x_0)}$$

$$\underline{\pi_1(V, x_0)} \rightarrow \Pi_1(V) \rightarrow \underline{\pi_1(V, x_0)}$$

$$\underline{\pi_1(U \cap V, x_0)} \rightarrow \Pi_1(U \cap V) \rightarrow \underline{\pi_1(U \cap V, x_0)}$$

are all identity functors.

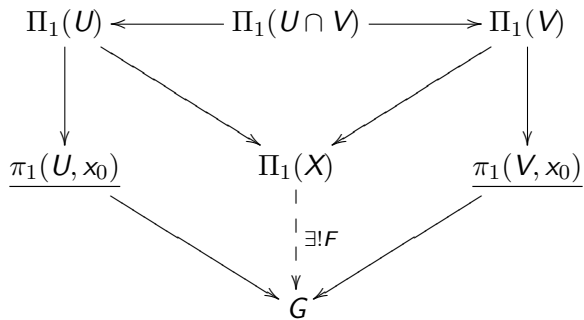


Suppose there is a group  $G$  that fits into the following commutative diagram:

$$\begin{array}{ccccc} \Pi_1(U) & \longleftarrow & \Pi_1(U \cap V) & \longrightarrow & \Pi_1(V) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\pi_1(U, x_0)} & \longleftarrow & \underline{\pi_1(U \cap V, x_0)} & \longrightarrow & \underline{\pi_1(V, x_0)} \\ & \searrow & \downarrow & \swarrow & \\ & & \underline{G} & & \end{array}$$



By Seifert-van Kampen Theorem (Groupoid version),





Thus, we obtain a morphism

$$\underline{\pi_1(X, x_0)} \hookrightarrow \Pi_1(X) \xrightarrow{F} \underline{G}$$

which fits into a commutative diagram

$$\begin{array}{ccc}
 \underline{\pi_1(U \cap V, x_0)} & \longrightarrow & \underline{\pi_1(U, x_0)} \\
 \downarrow & & \downarrow \\
 \underline{\pi(V, x_0)} & \longrightarrow & \underline{\pi(X, x_0)} \\
 & \searrow & \searrow \\
 & & \underline{G}
 \end{array}$$

This leads to the required diagram in the category Group.  $\square$

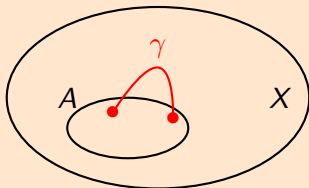




We also have the relative version.

## Definition

Let  $A \subset X$ , we define  $\Pi_1(X, A)$  be the full subcategory of  $\Pi_1(X)$  consists of objects in  $A$ .



For instance, when  $A = \{x_0\}$ , we have

$$\Pi_1(X, x_0) = \pi_1(X, x_0).$$



## Theorem

Let  $X = U \cup V$ ,  $U, V$  be open and  $A \subset X$  intersects each path connected components of  $U, V, U \cap V$ . Then we have a pushout

$$\begin{array}{ccc} \Pi_1(U \cap V, A) & \longrightarrow & \Pi_1(U, A) \\ \downarrow & & \downarrow \\ \Pi_1(V, A) & \longrightarrow & \Pi_1(X, A). \end{array}$$



## Example

For the Figure-8, which is  $S^1 \vee S^1$ .

$$S^1 \vee S^1 = \text{Figure-8}$$

It can be decomposed into  $U, V$  as follows

$$U = \text{Left circle} \quad V = \text{Right circle}$$

Since  $U, V$  are homotopic to  $S^1$ , and  $U \cap V$  is homotopic to a point, Seifert-van Kampen Theorem implies

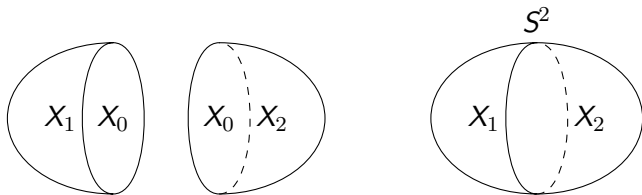
$$\pi_1(S^1 \vee S^1) = \pi_1(S^1) \star \pi_1(S^1) = \mathbb{Z} \star \mathbb{Z}.$$

In general, we have  $\pi_1(\bigvee_{i=1}^n S^1) = \underbrace{\mathbb{Z} \star \cdots \star \mathbb{Z}}_n$ .



## Example

Consider the 2-sphere  $S^2 = D_1 \cup D_2$  where  $D_i$  are open disks and  $D_0 = D_1 \cap D_2$  is an annulus.  $D_i$  is an open neighbourhood of  $X_i$ .



Since  $\pi_1(D_1) = \pi_1(D_2) = 1$ ,  $\pi_1(D_0) = \pi_1(S^1) = \mathbb{Z}$ , we deduce that

$$\pi_1(S^2) = (1 \star 1) / \mathbb{Z} = 1.$$

Similar argument shows that

$$\pi_1(S^n) = 1, \quad n \geq 2.$$



## Example

Let us identify  $X = S^1$  with the unit circle in  $\mathbb{R}^2$ . Consider

$$U = \{(x, y) \in S^1 \mid y > -1/2\}, \quad V = \{(x, y) \in S^1 \mid y < 1/2\}$$

and  $A = \{(\pm 1, 0)\}$ . Then we obtain a pushout

$$\begin{array}{ccc} \Pi_1(U \cap V, A) & \longrightarrow & \Pi_1(U, A) \\ \downarrow & & \downarrow \\ \Pi_1(V, A) & \longrightarrow & \Pi_1(S^1, A). \end{array}$$



This implies that the groupoid  $\Pi_1(S^1, A)$  contains two objects  $A_1 = (1, 0), A_2 = (-1, 0)$  with morphisms

$$\text{Hom}_{\Pi_1(S^1, A)}(A_1, A_1) = \{(\gamma_- \gamma_+)^n\}_{n \in \mathbb{Z}}$$

$$\text{Hom}_{\Pi_1(S^1, A)}(A_1, A_2) = \{(\gamma_+ \gamma_-)^n \gamma_+\}_{n \in \mathbb{Z}}$$

$$\text{Hom}_{\Pi_1(S^1, A)}(A_2, A_1) = \{(\gamma_- \gamma_+)^n \gamma_-\}_{n \in \mathbb{Z}}$$

$$\text{Hom}_{\Pi_1(S^1, A)}(A_2, A_2) = \{(\gamma_+ \gamma_-)^n\}_{n \in \mathbb{Z}}.$$

Here  $\gamma_+$  represent the semi-circle from  $(1, 0)$  to  $(-1, 0)$  anti-clockwise, and  $\gamma_-$  represent the semi-circle from  $(-1, 0)$  to  $(1, 0)$  anti-clockwise.

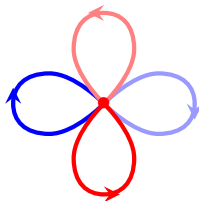
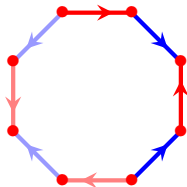


## Example

Consider the closed orientable surface  $\Sigma_g$  of genus  $g$ , which admits a polygon presentation

$$P = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Here is a figure for  $g = 2$ .





The edges of the polygon form  $V_{2g} = \bigvee_{i=1}^{2g} S^1$ .

Let  $U$  be the interior of the polygon and  $V$  be a small open neighbourhood of  $V_{2g}$ .  $U \cap V$  is an annulus, which is homotopy to  $S^1$  with generator  $P$  as above. Thus

$$\begin{aligned} \pi_1(\Sigma_g) &= \left( \prod_{i=1}^{2g} \mathbb{Z} \right) \star 0 / \mathbb{Z} \\ &= \langle a_i, b_i \mid i = 1, \dots, g \rangle / (a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}). \end{aligned}$$





## Example

Using the polygon presentation  $P = a^2$ , we can similarly compute

$$\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$$



## The Jordan Curve Theorem



We give an application of Seifert-van Kampen Theorem to prove the Jordan Curve Theorem. This is an example which sounds totally obvious intuitively, but turns out to be very difficult to prove rigorously.

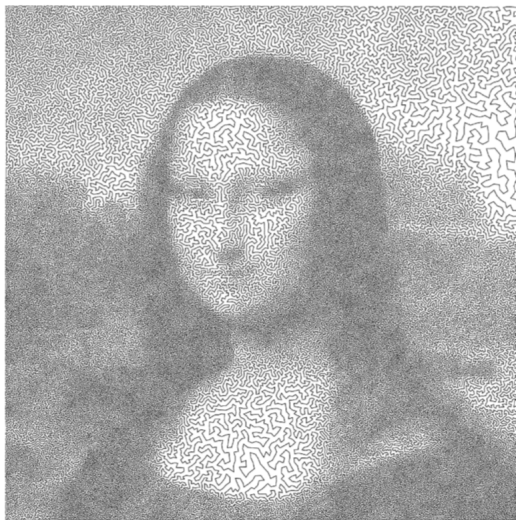


## Definition

A simple closed curve is a subset of  $\mathbb{R}^2$  (or  $S^2$ ) which is homeomorphic to the circle  $S^1$ .

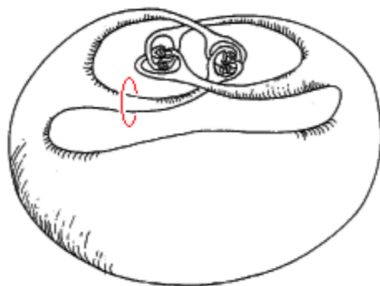
## Theorem (The Jordan Curve Theorem)

Let  $C$  be a simple closed curve in the sphere  $S^2$ . Then the complement of  $C$  has exactly two connected components.



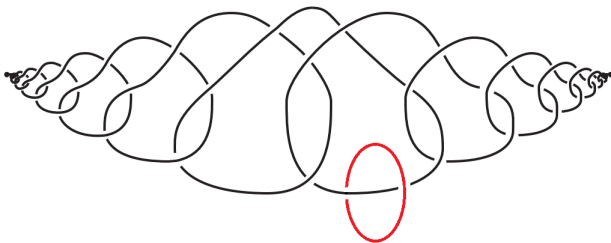


## Alexander's Horned Sphere





Wild arc in  $\mathbb{R}^3$





# Proof

We sketch a proof here. Since  $S^2$  is locally path connected, we would not distinguish connected and path connected here.

By an [arc](#), we mean a subset of  $S^2$  which is homeomorphic to the interval  $I$ . We first show

if  $A$  is an arc in  $S^2$ , then  $S^2 \setminus A$  is connected.





In fact, assume there are two points  $\{a, b\}$  which are disconnected in  $S^2 \setminus A$ . Let us subdivide  $A = A_1 \cup A_2$  into two intervals where  $A_1 = [0, 1/2]$ ,  $A_2 = [1/2, 1]$  using the homeomorphism  $A = [0, 1]$ .

We argue that  $a, b$  are disconnected in either  $S^2 \setminus A_1$  or  $S^2 \setminus A_2$ .

Let us choose a set  $D$  which contains one point from each connected component of  $S^2 \setminus A$  and such that  $\{a, b\} \subset D$ .

Apply Seifert-van Kampen Theorem to

$$V_1 = S^2 \setminus A_1, \quad V_2 = S^2 \setminus A_2, \quad V_1 \cap V_2 = S^2 \setminus A.$$



We obtain a pushout in Groupoid

$$\begin{array}{ccc}
 \Pi_1(V_1 \cap V_2, D) & \longrightarrow & \Pi_1(V_2, D) \\
 \downarrow & & \downarrow \\
 \Pi_1(V_1, D) & \longrightarrow & \Pi_1(Y, D).
 \end{array}$$

Here  $Y = V_1 \cup V_2$  is the complement of a point in  $S^2$ . If  $\{a, b\}$  are connected in both  $V_1$  and  $V_2$ , then the pushout implies that there exists a nontrivial morphism  $a \rightarrow a$  in  $\Pi_1(Y, D)$  by a composition

$$a \xrightarrow{\text{in } V_1} b \xrightarrow{\text{in } V_1 \cap V_2} b \xrightarrow{\text{in } V_2} a$$

But this can not true since  $Y$  is contractible. So let us assume  $a, b$  are disconnected in  $V_1 = S^2 \setminus A_1$ . Run the above process replacing  $A$  by  $A_1$ , and keep doing this, we end up with contradiction in the limit. This proves our claim above for the arc.



Secondly, we show

the complement of  $C$  in  $S^2$  is disconnected.

Otherwise, assume  $S^2 \setminus C$  is connected. Let us divide  $C = A_1 \cup A_2$  into two intervals  $A_1, A_2$  which intersect at two endpoints  $\{a, b\}$ .

$$U_1 = S^2 \setminus A_1, U_2 = S^2 \setminus A_2, U_1 \cap U_2 = S^2 \setminus C, X = U_1 \cup U_2 = S^2 \setminus \{a, b\}.$$

Since  $U_1, U_2, U_1 \cap U_2$  are all connected, Seifert-van Kampen Theorem leads to a pushout in Group

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2) & \longrightarrow & \pi_1(U_2) \\ \downarrow & & \downarrow \\ \pi_1(U_1) & \longrightarrow & \pi_1(X). \end{array}$$

Observe  $\pi_1(X) = \mathbb{Z}$ . We show both  $\pi_1(U_i) \rightarrow \pi_1(X)$  are trivial. This would lead to a contradiction.



Let us identify  $S^2 = \mathbb{R}^2 \cup \{\infty\}$  and assume  $a = 0, b = \infty$ , so  $A_1$  is parametrized by a path  $\alpha$  from 0 to  $\infty$ . Let  $\gamma$  be an arbitrary loop in  $U_1$ , we need to show  $\gamma$  becomes trivial in  $X$ . Let  $R > 0$  be sufficient large such that  $\gamma$  is contained in the ball of radius  $R$  centered at the origin in  $\mathbb{R}^2$ . Consider the homotopy

$$F(t, s) = \gamma(t) - \alpha(s), \quad \gamma_s := F(-, s).$$

We have  $\gamma_0 = \gamma$ . Assume  $\alpha(t_0) > R$ , then  $\gamma_{t_0}$  lies inside the ball of radius  $R$  centered at  $\alpha(t_0)$ , which is contractible in  $X$ . This implies  $\gamma$  is trivial in  $X$ . The same argument applies to  $A_2$ .



Finally, we show

the complement of  $C$  in  $S^2$  has exactly two connected components.

Let  $C = A_1 \cup A_2$  and  $U_1, U_2$  as in the previous step. We have a pushout in Groupoid

$$\begin{array}{ccc} \Pi_1(U_1 \cap U_2) & \longrightarrow & \Pi_1(U_2) \\ \downarrow & & \downarrow \\ \Pi_1(U_1) & \longrightarrow & \Pi_1(X). \end{array}$$



Assume  $S^2 \setminus C$  has at least three components, so we can choose  $p_1, p_2, p_3$  from three different connected components. Since  $U_1, U_2$  are connected, and  $p_i$ 's are disconnected in  $U_1 \cap U_2$ , the following two compositions

$$p_1 \xrightarrow{\text{in } U_1} p_2 \xrightarrow{\text{in } U_1 \cap U_2} p_2 \xrightarrow{\text{in } U_2} p_1, \quad p_1 \xrightarrow{\text{in } U_1} p_3 \xrightarrow{\text{in } U_1 \cap U_2} p_3 \xrightarrow{\text{in } U_2} p_1$$

give two free generators in  $\pi_1(X, p_1)$ .

But  $\pi_1(X, p_1) = \mathbb{Z}$ . This is a contradiction. □